

Independent Sets in Classes Related to Chair/Fork-free Graphs

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Abstract

The MAXIMUM WEIGHT INDEPENDENT SET (MWIS) problem on graphs with vertex weights asks for a set of pairwise nonadjacent vertices of maximum total weight. MWIS is known to be *NP*-complete in general, even under various restrictions. Let $S_{i,j,k}$ be the graph consisting of three induced paths of lengths i, j, k with a common initial vertex. The complexity of the MWIS problem for $S_{1,2,2}$ -free graphs, and for $S_{1,1,3}$ -free graphs are open. In this paper, we show that the MWIS problem can be solved in polynomial time for $(S_{1,2,2}, S_{1,1,3}, \text{co-chair})$ -free graphs, by analyzing the structure of the subclasses of this class of graphs. This extends some known results in the literature.

Keywords: Graph algorithms; Independent sets; Claw-free graphs; Chair-free graphs; Clique separators; Modular decomposition.

1 Introduction

For notation and terminology not defined here, we follow [6]. Let P_n and C_n denote respectively the path, and the cycle on n vertices. If \mathcal{F} is a family of graphs, a graph G is said to be \mathcal{F} -free if it contains no induced subgraph isomorphic to any graph in \mathcal{F} .

In a graph G , an *independent (or stable) set* is a subset of mutually non-adjacent vertices in G . The MAXIMUM INDEPENDENT SET (MIS) problem asks for an independent set of G with maximum cardinality. The MAXIMUM WEIGHT INDEPENDENT SET (MWIS) problem asks for an independent set

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of total maximum weight in the given graph G with vertex weight function w on $V(G)$. The M(W)IS problem is well known to be NP -complete in general and hard to approximate; it remains NP -complete even on restricted classes of graphs [9, 32]. On the other hand, the M(W)IS problem is known to be solvable in polynomial time on many graph classes such as: chordal graphs [12]; P_4 -free graphs [10]; perfect graphs [14]; $2K_2$ -free graphs [11]; claw-free graphs [28]; fork-free graphs [23]; apple-free graphs [7]; and P_5 -free graphs [20].

For integers $i, j, k \geq 0$, let $S_{i,j,k}$ denote a tree with exactly three vertices of degree one, being at distance i, j and k from the unique vertex of degree three. The graph $S_{0,1,2}$ is isomorphic to P_4 and the graph $S_{0,2,2}$ is isomorphic to P_5 . The graph $S_{1,1,1}$ is called a *claw* and $S_{1,1,2}$ is called a *chair* or *fork*. Also, note that $S_{i,j,k}$ is a subdivision of a claw, if $i, j, k \geq 1$.

Alekseev [1] showed that the M(W)IS problem remains NP -complete on H -free graphs, whenever H is connected, but neither a path nor a subdivision of the claw. As mentioned above, the complexity status of the MWIS problem in the graphs classes defined by a single forbidden induced subgraph of the form $S_{i,j,k}$ was solved for the case $i + j + k \leq 4$. However, for larger $i + j + k$, the complexity of MWIS in $S_{i,j,k}$ -free graphs is unknown. In particular, the class of P_6 -free graphs, the class of $S_{1,2,2}$ -free graphs, and the class of $S_{1,1,3}$ -free graphs constitute the minimal classes, defined by forbidding a single connected subgraph on six vertices, for which the computational complexity of M(W)IS problem is unknown. Also, it is known that there is an $n^{O(\log^2 n)}$ -time, polynomial-space algorithm for MWIS on P_6 -free graphs [21]. This implies that MWIS on P_6 -free graphs is not NP -complete, unless all problems in NP can be solved in quasi-polynomial time. On the other hand, MWIS is shown to be solvable in polynomial time for several subclasses of $S_{i,j,k}$ -free graphs, for $i + j + k \geq 5$ such as: $(P_6, \text{triangle})$ -free graphs [5]; $(P_6, K_{1,p})$ -free graphs [27]; (P_6, C_4) -free [2, 29]; $(P_6, \text{diamond})$ -free graphs [30]; (P_6, banner) -free graphs [16]; $(P_6, \text{co-banner})$ -free graphs [31]; $(P_6, S_{1,2,2}, \text{co-chair})$ -free graphs [19]; $(S_{1,1,3}, \text{banner})$ -free graphs [18]; and $(S_{1,2,2}, \text{bull})$ -free graphs [18]. It is also known that the MIS problem can be solved in polynomial time for some subclasses of $S_{i,j,k}$ -free graphs such as: $S_{1,2,k}$ -free planar graphs and $S_{1,k,k}$ -free graphs of low degree [22], and $S_{2,2,2}$ -free sub-cubic graphs [25]; and see [13, Table 1] for several other subclasses. See Figure 1 for some of the special graphs used in this paper.

In this paper, we show that the MWIS problem can be efficiently solved in the class of $(S_{1,2,2}, S_{1,1,3}, \text{co-chair})$ -free graphs by analyzing the structure of the subclasses of this class of graphs. This result extends some known results

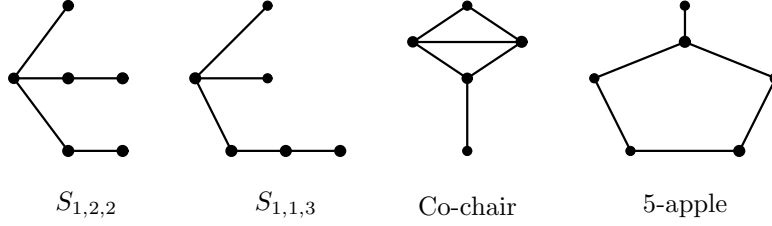


Figure 1: Some special graphs.

in the literature such as: the aforementioned results for P_4 -free graphs, and $(P_5, \text{co-chair})$ -free graphs [15]. A preliminary version (extended abstract) of this paper is appearing in [17].

2 Notation and terminology

Let G be a finite, undirected and simple graph with vertex-set $V(G)$ and edge-set $E(G)$. We let $|V(G)| = n$ and $|E(G)| = m$. For a vertex $v \in V(G)$, the *neighborhood* $N(v)$ of v is the set $\{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N[v]$ is the set $N(v) \cup \{v\}$. The neighborhood $N(X)$ of a subset $X \subseteq V(G)$ is the set $\{u \in V(G) \setminus X : u \text{ is adjacent to a vertex of } X\}$. Given a subgraph H of G and $v \in V(G) \setminus V(H)$, let $N_H(v)$ denote the set $N(v) \cap V(H)$, and for $X \subseteq V(G) \setminus V(H)$, let $N_H(X)$ denote the set $N(X) \cap V(H)$. For any two subsets $S, T \subseteq V(G)$, we say that S is *complete* to T if every vertex in S is adjacent to every vertex in T .

A *hole* is a chordless cycle C_k , where $k \geq 5$. An *odd hole* is a hole C_{2k+1} , where $k \geq 2$.

The k -*apple* is the graph obtained from a chordless cycle C_k of length $k \geq 4$ by adding a vertex that has exactly one neighbor on the cycle.

The *diamond* is the graph $K_4 - e$ with vertex-set $\{v_1, v_2, v_3, v_4\}$ and edge-set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$.

The *co-chair* is the graph with vertex-set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge-set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3, v_4v_5\}$; it is the complement graph of the *chair/fork* graph (see Figure 1).

A vertex $z \in V(G)$ *distinguishes* two other vertices $x, y \in V(G)$ if z is adjacent to one of them and nonadjacent to the other. A set $M \subseteq V(G)$ is a *module* in G if no vertex from $V(G) \setminus M$ distinguishes two vertices from M . The *trivial modules* in G are $V(G)$, \emptyset , and all one-vertex sets. A graph G is *prime* if it contains only trivial modules. Note that prime graphs on at least three vertices are connected.

A class of graphs \mathcal{G} is *hereditary* if every induced subgraph of a member of \mathcal{G} is also in \mathcal{G} . We will use the following theorem by Lözin and Milanič [23].

Theorem 1 ([23]). *Let \mathcal{G} be a hereditary class of graphs. If there is a constant $p \geq 1$ such that the MWIS problem can be solved in time $O(|V(G)|^p)$ for every prime graph G in \mathcal{G} , then the MWIS problem can be solved in time $O(|V(G)|^p + |E(G)|)$ for every graph G in \mathcal{G} .* \square

Let \mathcal{C} be a class of graphs. A graph G is *nearly \mathcal{C}* if for every vertex v in $V(G)$ the graph induced by $V(G) \setminus N[v]$ is in \mathcal{C} . Let $\alpha_w(G)$ denote the weighted independence number of G . Obviously, we have:

$$\alpha_w(G) = \max\{w(v) + \alpha_w(G \setminus N[v]) \mid v \in V(G)\}. \quad (1)$$

Thus, whenever MWIS is solvable in time T on a class \mathcal{C} , then it is solvable on nearly \mathcal{C} graphs in time $n \cdot T$.

A *clique* in G is a subset of pairwise adjacent vertices in G . A *clique separator* (or *clique cutset*) in a connected graph G is a subset Q of vertices in G which induces a complete graph, such that the graph induced by $V(G) \setminus Q$ is disconnected. A graph is an *atom* if it does not contain a clique separator.

We will also use the following theorem given in [18].

Theorem 2 ([18]). *Let \mathcal{C} be a class of graphs such that MWIS can be solved in time $O(f(n))$ for every graph in \mathcal{C} with n vertices. Then in any hereditary class of graphs whose all atoms are nearly \mathcal{C} the MWIS problem can be solved in time $O(n^2 \cdot f(n))$.* \square

The following notation will be used several times in the proofs. Given a graph G , let v be a vertex in G and H be an induced subgraph of $G \setminus \{v\}$ such that v has no neighbor in H . Let $t = |V(H)|$. Then we define the following sets:

$$\begin{aligned} Q &= \text{the component of } G \setminus (V(H) \cup N(V(H))) \text{ that contains } v, \\ A_i &= \{x \in V(G) \setminus V(H) \mid |N_H(x)| = i\} \quad (1 \leq i \leq t), \\ A_i^+ &= \{x \in A_i \mid N(x) \cap Q \neq \emptyset\}, \\ A_i^- &= \{x \in A_i \mid N(x) \cap Q = \emptyset\}, \\ A^+ &= A_1^+ \cup \dots \cup A_t^+ \text{ and } A^- = A_1^- \cup \dots \cup A_t^-. \end{aligned}$$

So, $N(H) = A^+ \cup A^-$. Note that, by the definition of Q and A^+ , we have $A^+ = N(Q)$. Hence A^+ is a separator between H and Q in G .

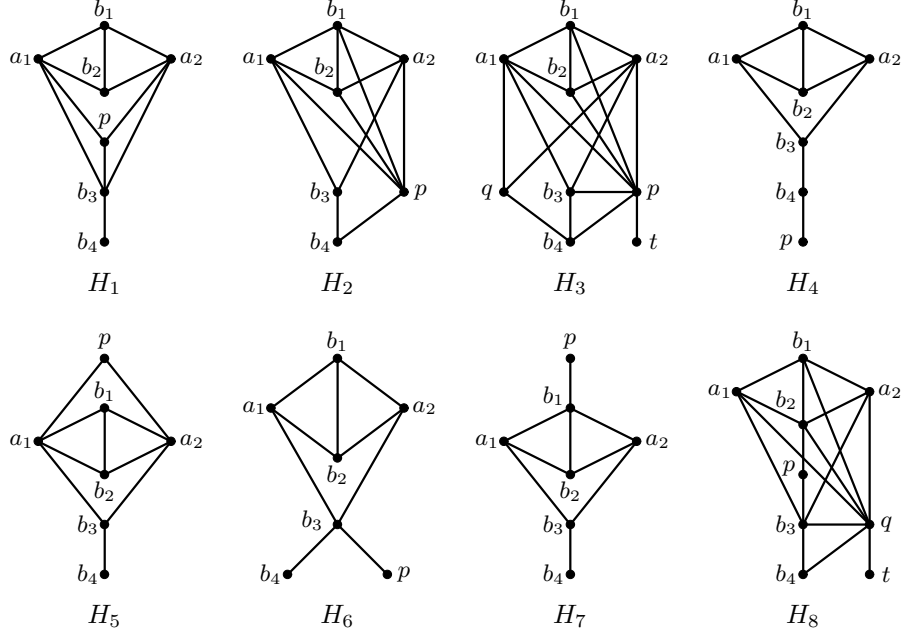


Figure 2: Graphs H_i , $i \in \{1, 2, \dots, 8\}$ used in Lemma 1.

3 Preliminary lemmas

Lemma 1. *Let $G = (V, E)$ be a prime co-chair-free graph. Then G is (H_1, H_2, H_3) -free. Further, if G is $S_{1,2,2}$ -free, then G is (H_4, H_5) -free, and if G is $S_{1,1,3}$ -free, then G is (H_4, H_6, H_7, H_8) -free. See Figure 2 for the graphs H_i , $i \in \{1, 2, \dots, 8\}$.*

Proof. Suppose to the contrary that G contains an induced H_i , for some $i \in \{1, 2, \dots, 8\}$ (as shown in Figure 2). Since G is prime, $\{a_1, a_2\}$ is not a module in G , so there exists a vertex $x \in V \setminus V(H_i)$ such that (up to symmetry) $xa_1 \in E$ and $xa_2 \notin E(G)$. Then since $\{x, a_1, b_1, b_2, a_2\}$ does not induce a co-chair in G , x is adjacent to one of b_1, b_2 .

Suppose that x is adjacent to both of b_1, b_2 . Then since $\{x, b_1, b_2, a_2, b_3\}$ does not induce a co-chair in G , $xb_3 \in E$, and since $\{x, b_1, b_2, a_2, b_4\}$ does not induce a co-chair in G , $xb_4 \notin E$. But, now $\{x, a_1, b_1, b_3, b_4\}$ induces a co-chair in G , which is a contradiction. Therefore, x is adjacent to exactly one of b_1, b_2 .

Suppose that $i \neq 7, 8$. We may assume (up to symmetry) that $xb_1 \in E$ and $xb_2 \notin E$. Since $\{x, a_1, b_1, b_2, b_4\}$ does not induce a co-chair in G , $xb_4 \notin E$.

E , and then since $\{x, a_1, b_1, b_3, b_4\}$ does not induce a co-chair in G , $xb_3 \notin E$.

Now, we prove a contradiction as follows:

$i = 1$: Since $\{x, a_1, a_2, b_3, p\}$ does not induce a co-chair in G , $xp \in E$. But, now $\{x, a_1, b_3, b_4, p\}$ induces a co-chair in G , which is a contradiction. Thus, G is H_1 -free.

$i = 2$: Since $\{x, a_1, b_1, p, b_4\}$ does not induce a co-chair in G , $xp \in E$. But, now $\{x, b_1, p, a_2, b_3\}$ induces a co-chair in G , which is a contradiction. Thus, G is H_2 -free.

$i = 3$: Since $\{x, a_1, b_1, b_2, t\}$ does not induce a co-chair in G , $xt \notin E$, and then since $\{x, a_1, b_1, p, t\}$ does not induce a co-chair in G , $xp \in E$. Then since $\{x, b_1, p, a_2, q\}$ does not induce a co-chair in G , $xq \in E$. But, now $\{b_1, x, a_1, q, b_4\}$ induces a co-chair in G , which is a contradiction. Thus, G is H_3 -free.

$i = 4$: Since $\{x, a_1, b_1, b_2, p\}$ does not induce a co-chair in G , $xp \notin E$. But, now $\{x, a_1, b_3, b_4, p, a_2\}$ induces an $S_{1,2,2}$ in G or $\{x, a_1, b_3, b_4, p, b_2\}$ induces an $S_{1,1,3}$ in G , a contradiction. Thus, G is H_4 -free.

$i = 5$: Since $\{x, b_1, a_2, b_3, b_4, p\}$ does not induce an $S_{1,2,2}$ in G , $xp \in E$. But, now $\{x, p, a_2, b_3, b_4, b_2\}$ induces an $S_{1,2,2}$ in G , which is a contradiction. Thus, G is H_5 -free.

$i = 6$: Since $\{x, a_1, b_1, b_2, p\}$ does not induce a co-chair in G , $xp \notin E$. But, now $\{x, b_1, a_2, b_3, b_4, p\}$ induces an $S_{1,1,3}$ in G , which is a contradiction. Thus, G is H_6 -free.

Suppose that $i = 7$. Note that x is adjacent to exactly one of b_1, b_2 . Then as earlier $xb_3, xb_4 \notin E$ (otherwise, G induces a co-chair). Now, if $xb_1 \in E$ and $xb_2 \notin E$, then since $\{x, b_1, a_2, b_3, b_4, p\}$ does not induce an $S_{1,1,3}$ in G , $xp \in E$. But, now $\{p, x, b_1, a_1, b_3\}$ induces a co-chair in G , which is a contradiction. Next, if $xb_2 \in E$ and $xb_1 \notin E$, then since $\{x, a_1, b_2, b_1, p\}$ does not induce a co-chair in G , $xp \in E$. But, now $\{b_4, b_3, a_1, x, p, a_2\}$ induces an $S_{1,1,3}$ in G , which is a contradiction. Thus, G is H_7 -free.

Suppose that $i = 8$. Again as earlier $xb_3, xb_4 \notin E$ (otherwise, G induces a co-chair). Now, if $xb_1 \in E$ and $xb_2 \notin E$, then since $\{x, a_1, b_1, b_2, p\}$ does not induce a co-chair in G , $xp \in E$. Also, since $\{x, a_1, b_1, b_2, t\}$ does not induce a co-chair in G , $xt \notin E$, and then since $\{x, a_1, b_1, q, t\}$ does not induce a co-chair in G , $xq \in E$. But, now $\{a_2, b_1, q, x, p\}$ induces a co-chair in G , which is a contradiction. Next, if $xb_2 \in E$ and $xb_1 \notin E$, then since $\{x, b_2, p, b_3, b_4, b_1\}$ does not induce an $S_{1,1,3}$ in G , $xp \in E$. Also, since $\{x, a_1, b_1, b_2, t\}$ does not induce a co-chair in G , $xt \notin E$, and then since $\{x, a_1, b_2, q, t\}$ does not induce a co-chair in G , $xq \in E$. But, now $\{p, x, b_2, q, t\}$ induces a co-chair in G , which is a contradiction. Thus, G is H_8 -free.

This completes the proof of Lemma 1. \square

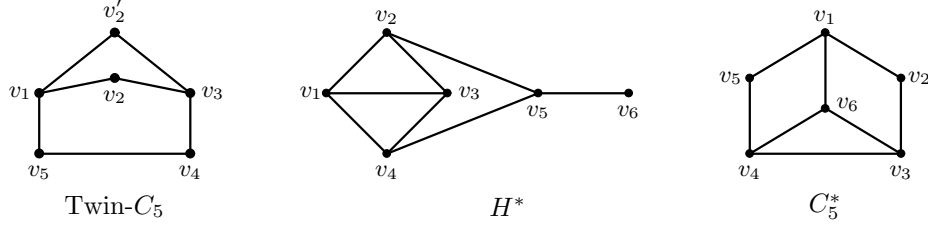


Figure 3: Graphs $\text{twin-}C_5$, H^* and C_5^* .

Lemma 2. *If $G = (V, E)$ is a prime (5-apple, C_5^* , diamond)-free graph, then G is $\text{twin-}C_5$ -free.*

Proof. Suppose to the contrary that G contains an induced $\text{twin-}C_5$, say H as shown in Figure 3. Since G is prime, $\{v_2, v_2'\}$ is not a module in G , so there exists a vertex x in $V \setminus V(H)$ such that (up to symmetry) $xv_2' \in E$ and $xv_2 \notin E$. Then since $\{v_2', v_1, v_3, v_4, v_5, x\}$ does not induce a 5-apple in G , $xv_i \in E$, for some $i \in \{1, 3, 4, 5\}$. If $xv_1 \in E$, then since G is diamond-free, $xv_3, xv_5 \notin E$. Then since $\{v_1, v_2, v_3, v_4, v_5, x\}$ does not induce a 5-apple in G , $xv_4 \in E$, but then $\{v_1, v_2', v_3, v_4, v_5, x\}$ induces a C_5^* in G , which is a contradiction. A similar contradiction arises if we assume $xv_3 \in E$. So, we may assume that $xv_1, xv_3 \notin E$. Then since G is 5-apple-free, $xv_4 \in E$ and $xv_5 \in E$. Now, $\{v_1, v_2', v_3, v_4, v_5, x\}$ induces a C_5^* in G , which is a contradiction. So, G is $\text{twin-}C_5$ -free. \square

Lemma 3 ([15]). *If $G = (V, E)$ is a prime (co-chair, gem)-free graph, then G is diamond-free.*

4 $(S_{1,2,2}, S_{1,1,3}, \text{diamond})$ -free graphs

In this section, we show that the MWIS problem can be efficiently solved in the class of $(S_{1,2,2}, S_{1,1,3}, \text{diamond})$ -free graphs by analyzing the atomic structure of the subclasses of this class of graphs.

4.1 $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple}, C_5^*)$ -free graphs

Theorem 3. *Let $G = (V, E)$ be a prime $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple}, C_5^*)$ -free graph. If G contains an odd hole C_{2k+1} with $k \geq 2$, then G is claw-free.*

Proof. Since G is prime, it is connected, and by Lemma 2, G is twin- C_5 -free. Let C denotes a shortest odd hole C_{2k+1} in G with vertices $v_1, v_2, \dots, v_{2k+1}$ and edges $v_i v_{i+1}, v_{2k+1} v_1 \in E$, where $i \in \{1, 2, \dots, 2k\}$ with $k \geq 2$. Then it is verified that the following claim holds.

Claim 3.1. *If $x \in V(G) \setminus V(C)$ has a neighbor on C , then there exists an i such that $N(x) \cap V(C) = \{v_i, v_{i+1}\}$.*

Proof of Claim 3.1: If $k = 2$, since G is (5-apple, C_5^* , twin- C_5 , diamond)-free, the claim holds. So, suppose that $k \geq 3$. To prove the claim, we prove the following:

- (1) There exists an i such that $xv_i, xv_{i+1} \in E$ and $xv_{i-1}, xv_{i+2} \notin E$.
- (2) Either $|N(x) \cap V(C)| = 2$ or $|N(x) \cap V(C)| = 4$. Moreover, there exists an i such that $N(x) \cap V(C) = \{v_i, v_{i+1}\}$ (if $|N(x) \cap V(C)| = 2$), and $N(x) \cap V(C) = \{v_i, v_{i+1}, v_j, v_{j+1}\}$, for some $j \in \{i+3, i+4, \dots, i+2k-2\}$ (if $|N(x) \cap V(C)| = 4$).

Since x has a neighbor on C , we may assume that x is adjacent to v_i on C . If (1) does not hold, then $xv_{i+1}, xv_{i-1} \notin E$. Then since $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, x\}$ does not induce an $S_{1,2,2}$ in G , we have either $xv_{i-2} \in E$ or $xv_{i+2} \in E$. We may assume, up to symmetry, that $xv_{i-2} \in E$. Then since $\{v_{i+3}, v_{i+2}, v_{i+1}, v_i, v_{i-1}, x\}$ does not induce a C_5 or an $S_{1,1,3}$ in G , we have $xv_{i+2} \in E$. Then since $\{v_{i+1}, v_{i+2}, x, v_{i-2}, v_{i-1}, v_{i-3}\}$ does not induce an $S_{1,1,3}$ in G , $xv_{i-3} \in E$. Then since G is diamond-free, $\{v_{i+3}, v_{i-3}, x, v_i, v_{i+1}, v_{i-1}\}$ induces an $S_{1,1,3}$ in G (if $k = 3$) or $\{v_{i-4}, v_{i-3}, x, v_i, v_{i+1}, v_{i-1}\}$ induces an $S_{1,1,3}$ in G (if $k \geq 4$), a contradiction. So (1) holds.

By (1), we have $\{v_i, v_{i+1}\} \subseteq N(x) \cap V(C)$, and $xv_{i-1}, xv_{i+2} \notin E$. Further, if there exists an index $j \in \{i+3, i+4, \dots, i+2k-1\}$ such that $xv_j \in E$ and $xv_{j-1} \notin E$, then $xv_{j+1} \in E$ (for, otherwise, $\{v_{i-1}, v_i, v_{i+1}, v_{i+2}, x\} \cup \{v_{j-1}, v_j, v_{j+1}\}$ induces an $S_{1,1,3}$ in G). Now, if x is adjacent to a vertex v_t on C , where $t \notin \{i-1, i, i+1, i+2, j-1, j, j+1, j+2\}$, then either a diamond or an $S_{1,2,2}$ is an induced subgraph of G , which is a contradiction. Hence $N(x) \cap V(C) = \{v_i, v_{i+1}\}$ or $N(x) \cap V(C) = \{v_i, v_{i+1}, v_j, v_{j+1}\}$, for some $j \in \{i+3, i+4, \dots, i+2k-2\}$. So (2) holds.

Further, if $|N(x) \cap V(C)| = 4$, then G contains an odd hole C' shorter than C , which is a contradiction to the choice of C . \diamond

To prove the theorem, we suppose for contradiction that G contains an induced claw, say K with vertex-set $\{a, b, c, d\}$ and edge-set $\{ab, ac, ad\}$. By Claim 3.1, K cannot have more than two vertices on C . Also, at most one vertex in $\{b, c, d\}$ belongs to C . Now we have following cases (the other cases are symmetric):

- (1) $V(K) \cap V(C) = \{a, d\}$: Let y be the other neighbor of a on C . Then by Claim 3.1, $by, cy \in E$. But, now $\{a, b, c, y\}$ induces a diamond in G , which is a contradiction.
- (2) $V(K) \cap V(C) = \{a\}$: We may assume (wlog.) that $a = v_1$. Then by Claim 3.1, at least two vertices in $\{b, c, d\}$ are adjacent either to v_2 or to v_{2k+1} , say b and c are adjacent to v_2 . Then $\{a, v_2, b, c\}$ induces a diamond in G , which is a contradiction.
- (3) $V(K) \cap V(C) = \{d\}$: We may assume (wlog.) that $d = v_1$. Then by Claim 3.1, up to symmetry, we may assume $av_2 \in E$. Suppose that $k = 2$. Then since G is $(S_{1,1,3}, 5\text{-apple}, \text{diamond})$ -free, both b and c have neighbors in C . To avoid a diamond in G and by Claim 3.1, we assume (wlog.) that $bv_3, bv_4, cv_4, cv_5 \in E$. But, now $\{d = v_1, a, b, v_4, v_5, c\}$ induces a C_5^* in G , which is a contradiction. So, suppose that $k \geq 3$. Then since G is diamond-free, $bv_2, cv_2 \notin E$. We claim that either $bv_3 \in E$ or $cv_3 \in E$. Otherwise, since $\{v_4, v_3, v_2, a, b, c\}$ does not induce an $S_{1,1,3}$ in G , either $bv_4 \in E$ or $cv_4 \in E$. But, now $\{v_4, v_3, v_2, a, b, c\}$ induces a C_5 in G , which is a contradiction to the fact that $k \geq 3$ and the choice of C . Thus, we may assume that $bv_3 \in E$. Then by Claim 3.1, $bv_4 \in E$. Then $cv_3 \notin E$ (for, otherwise, by Claim 3.1, $cv_4 \in E$, but then $\{v_3, v_4, b, c\}$ induces a diamond in G), and hence $cv_4 \notin E$ (for, otherwise, $\{a, c, v_4, v_3, v_2\}$ induces a C_5 in G). Now, $\{v_5, v_4, b, a, c\}$ induces a C_5 in G (if $cv_5 \in E$) or $\{v_5, v_4, b, a, d(=v_1), c\}$ induces an $S_{1,1,3}$ in G (if $cv_5 \notin E$), a contradiction.
- (4) $V(K) \cap V(C) = \emptyset$ and a vertex of K has a neighbor on C : Assume a has neighbors on C , say v_1 and v_2 . Then to avoid an induced claw intersecting C , both v_1 and v_2 have exactly two neighbors among b, c, d . We may assume (wlog.) that v_1 is adjacent to b and c . But, now $\{v_1, a, b, c\}$ induces a diamond in G , which is a contradiction. So, assume that a has no neighbor on C . Assume (wlog.) that b has a neighbor on C . By Claim 3.1, we may assume that $N(b) \cap V(C) = \{v_1, v_2\}$. Then since G is $S_{1,1,3}$ -free, both c and d have neighbors on C . Now, v_2 is adjacent to either c or d (for, otherwise, $\{v_3, v_2, b, a, c, d\}$ will induce an $S_{1,1,3}$ or a C_5 in G). Assume that $cv_2 \in E$. Since G is diamond-free, $cv_1 \notin E$ and by Claim 1, $cv_3 \in E$. Thus, $N(c) \cap V(C) = \{v_2, v_3\}$. By similar arguments, we see that $N(d) \cap V(C) = \{v_1, v_{2k+1}\}$. Now, $\{v_4, v_3, c, a, b, d\}$ induces an $S_{1,1,3}$ in G , which is a contradiction.
- (5) $V(K) \cap V(C) = \emptyset$ and no vertex of K has a neighbor on C : Since G

is connected, there exists an $i \in \{1, 2, \dots, 2k+1\}$ and a path $v_i = u_1 - u_2 - \dots - u_t - a$, say P connecting v_i and a in G (where $t \geq 2$) and with u_2 has a neighbor on C . By the choice of P , no vertex of this path has a neighbor on C except u_2 . By Claim 3.1, either $u_2 v_{i+1} \notin E$ or $u_2 v_{i-1} \notin E$. Assume that $u_2 v_{i+1} \notin E$. Now, $u_t \neq b, c, d$ (for, otherwise (wlog.) if $u_t = b$, then $\{v_{i+1}, v_i = u_1, u_2, \dots, u_t = b, a, c, d\}$ induces an $S_{1,1,3}$ in G). Then since G is diamond-free, at least two vertices in $\{b, c, d\}$ are not adjacent to u_t , say b and c . Then $\{v_{i+1}, v_i = u_1, u_2, \dots, u_t, a, b, c\}$ induces an $S_{1,1,3}$ in G , which is a contradiction.

Hence G is claw-free, and this completes the proof of the theorem. \square

Theorem 4. *The MWIS problem can be solved in polynomial time for $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple}, C_5^*)$ -free graphs.*

Proof. Let G be an $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple}, C_5^*)$ -free graph. If G is odd-hole-free, then G is (odd-hole, diamond)-free. Since MWIS in (odd-hole, diamond)-free graphs can be solved in polynomial time [8], MWIS can be solved in polynomial time for G . Suppose that G is prime and contains an odd-hole. Then by Theorem 3, G is claw-free. Since MWIS in claw-free graphs can be solved in polynomial time [28], MWIS can be solved in polynomial time for G . Then the time complexity is the same when G is not prime, by Theorem 1. \square

4.2 $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple})$ -free graphs

Theorem 5. *Let $G = (V, E)$ be an $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple})$ -free graph. Then G is nearly C_5^* -free.*

Proof. Let us assume on the contrary that there is a vertex $v \in V(G)$ such that $G \setminus N[v]$ contains an induced C_5^* , say H , with vertices named as in Figure 3. Let C denotes the 5-cycle induced by the vertices $\{v_1, v_2, v_3, v_4, v_5\}$ in H . For $i \in \{1, 2, \dots, 6\}$, we define sets A_i , A_i^+ , A^+ , and Q as in the last paragraph of Section 2. To prove the theorem, it is enough to show that $A^+ = \emptyset$. Assume to the contrary that $A^+ \neq \emptyset$, and let $x \in A^+$. Then there exists a vertex $z \in Q$ such that $xz \in E$. Then since G is (5-apple, diamond)-free, $|N_H(x) \cap V(C)| \in \{0, 2, 3\}$. Now:

- (i) If $|N_H(x) \cap V(C)| = 0$, then since $x \in N(H)$, $xv_6 \in E$. But then $\{z, x, v_6, v_1, v_2, v_5\}$ induces an $S_{1,1,3}$ in G , which is a contradiction.

- (ii) If $|N_H(x) \cap V(C)| = 2$, and if $N_H(x) \cap V(C) = \{v_i, v_{i+2}\}$, for some $i \in \{1, 2, 3, 4, 5\}$, $i \bmod 5$, then $\{z, x, v_{i+2}, v_{i+3}, v_{i+4}, v_i\}$ induces a 5-apple in G , which is a contradiction.
- (iii) If $|N_H(x) \cap V(C)| = 2$, and if $N_H(x) \cap V(C) = \{v_i, v_{i+1}\}$, for some $i \in \{1, 2, 3, 4, 5\}$, $i \bmod 5$, then since $\{z, x\} \cup V(H)$ does not induce a diamond or an $S_{1,1,3}$ in G , we have $i \neq 3$. Again, since G is diamond-free, $xv_6 \notin E$. But, then $\{z, x\} \cup V(H)$ induces either an $S_{1,1,3}$ or an $S_{1,2,2}$ in G , which is a contradiction.
- (iv) If $|N_H(x) \cap V(C)| = 3$, then since G is diamond-free, $N_H(x) \cap V(C) = \{v_i, v_{i+1}, v_{i+3}\}$, for some $i \in \{1, 2, 3, 4, 5\}$, $i \bmod 5$. Then since G is diamond-free, $i \neq 3$ and $xv_6 \notin E$. But, then $\{z, x\} \cup V(H)$ induces either an $S_{1,1,3}$ or an $S_{1,2,2}$ in G , which is a contradiction.

These contradictions show that $A^+ = \emptyset$, and hence G is nearly C_5^* -free. \square

Theorem 6. *The MWIS problem can be solved in polynomial time for $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple})$ -free graphs.*

Proof. Let G be an $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple})$ -free graph. Then by Theorem 5, G is nearly C_5^* -free. Since MWIS in $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple}, C_5^*)$ -free graphs can be solved in polynomial time (by Theorem 4), by the consequence given below equation (1) in Section 2, MWIS in $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple})$ -free graphs can be solved in polynomial time. \square

4.3 $(S_{1,2,2}, S_{1,1,3}, \text{diamond})$ -free graphs

Theorem 7. *Let $G = (V, E)$ be an $(S_{1,2,2}, S_{1,1,3}, \text{diamond})$ -free graph. Then every atom of G is nearly 5-apple-free.*

Proof. Let G' be an atom of G . We want to show that G' is nearly 5-apple-free, so let us assume on the contrary that there is a vertex $v \in V(G')$ such that $G' \setminus N[v]$ contains an induced 5-apple H . Let H have vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_1v_6\}$. Let C denotes the 5-cycle induced by the vertices $\{v_1, v_2, v_3, v_4, v_5\}$ in H . For $i \in \{1, 2, \dots, 6\}$, we define sets A_i, A_i^+, A^+ , and Q , with respect to G, v and H , as in the last paragraph of Section 2. Then, we immediately have the following:

Claim 7.1. *If $x \in N(H)$, then $|N_H(x) \cap V(C)| \leq 3$. In particular, if $x \in A^+$, then $|N_H(x) \cap V(C)| = 3$, and hence there exists an index $j \in \{1, 2, \dots, 5\}$, $j \bmod 5$, such that $N_H(x) \cap V(C) = \{v_j, v_{j+1}, v_{j+3}\}$. \diamond*

So, we have:

Claim 7.2. $A_1^+ = A_2^+ = A_5^+ = A_6^+ = \emptyset$.

Claim 7.3. If $x \in A_3^+$, then $N_H(x)$ is equal to $\{v_1, v_3, v_4\}$.

Proof of Claim 7.3. Suppose not. Then by Claim 7.1, there exists an index $j \in \{1, 2, 4, 5\}$, $j \bmod 5$, such that $N_H(x) = \{v_j, v_{j+1}, v_{j+3}\}$. Since $x \in A_3^+$, $xv_6 \notin E$, and there exists a vertex z in Q such that $xz \in E$. Now, if $N_H(x) = \{v_1, v_2, v_4\}$, then $\{v_6, v_1, x, v_4, v_3, z\}$ induces an $S_{1,2,2}$ in G , and if $N_H(x) = \{v_2, v_3, v_5\}$, then $\{v_6, v_1, v_5, x, v_3, z\}$ induces an $S_{1,1,3}$ in G , a contradiction. Since the other cases are symmetric, the claim follows. \diamond

Claim 7.4. $|A_3^+| = 1$.

Proof of Claim 7.4. Suppose not. Let $x, y \in A_3^+$. By Claim 7.3, $N_H(x) = N_H(y) = \{v_1, v_3, v_4\}$. Now, if $xy \in E$, then $\{v_4, x, y, v_1\}$ induces a diamond in G , and if $xy \notin E$, then $\{v_4, x, y, v_3\}$ induces a diamond in G , a contradiction. \diamond

Claim 7.5. If $x \in A_4^+$, then $N_H(x)$ is equal to $\{v_1, v_3, v_4, v_6\}$.

Proof of Claim 7.5. Suppose not. Then by Claim 7.1, there exists an index $j \in \{1, 2, 4, 5\}$, $j \bmod 5$, such that $N_H(x) \cap V(C) = \{v_j, v_{j+1}, v_{j+3}\}$. Since $x \in A_4^+$, $xv_6 \in E$ and there exists a vertex z in Q such that $xz \in E$. Now, if $N_H(x) \cap V(C) = \{v_1, v_2, v_4\}$, then $\{v_6, v_1, v_2, x\}$ induces a diamond in G , and if $N_H(x) \cap V(C) = \{v_2, v_3, v_5\}$, then $\{v_1, v_6, x, v_3, v_4, z\}$ induces an $S_{1,2,2}$ in G , a contradiction. Since the other cases are symmetric, the claim follows. \diamond

Claim 7.6. $|A_4^+| = 1$.

Proof of Claim 7.6. Suppose not. Let $x, y \in A_4^+$. By Claim 7.5, $N_H(x) = N_H(y) = \{v_1, v_3, v_4, v_6\}$. Now, if $xy \in E$, then $\{v_4, x, y, v_3\}$ induces a diamond in G , a contradiction and if $xy \notin E$, then $\{v_3, v_4, x, y\}$ induces a diamond in G , a contradiction. \diamond

Claim 7.7. At most one of A_3^+ or A_4^+ is non-empty.

Proof of Claim 7.7. Suppose not. Let $x \in A_3^+$ and $y \in A_4^+$. By Claim 7.3, $N_H(x) = \{v_1, v_3, v_4\}$, and by Claim 7.5, $N_H(y) = \{v_1, v_3, v_4, v_6\}$. Now, if $xy \in E$, then $\{v_3, x, y, v_1\}$ induces a diamond in G , a contradiction, and if $xy \notin E$, then $\{v_3, v_4, x, y\}$ induces a diamond in G , a contradiction. \diamond

Now, by Claim 7.2, $A^+ = A_3^+ \cup A_4^+$, and by Claims 7.4, 7.6 and 7.7, A^+ is a clique. Since A^+ is a separator between H and Q in G , we obtain that

$V(G') \cap A^+$ is a clique separator in G' between H and $V(G') \cap Q$ (which contains v). This is a contradiction to the fact that G' is an atom. \square

Theorem 8. *The MWIS problem can be solved in polynomial time for $(S_{1,2,2}, S_{1,1,3}, \text{diamond})$ -free graphs.*

Proof. Let G be an $(S_{1,2,2}, S_{1,1,3}, \text{diamond})$ -free graph. Then by Theorem 7, every atom of G is nearly 5-apple-free. Since MWIS in $(S_{1,2,2}, S_{1,1,3}, \text{diamond}, 5\text{-apple})$ -free graphs can be solved in polynomial time (by Theorem 6), MWIS in $(S_{1,2,2}, S_{1,1,3}, \text{diamond})$ -free graphs can be solved in polynomial time, by Theorem 2. \square

5 $(S_{1,2,2}, S_{1,1,3}, \text{co-chair})$ -free graphs

In this section, we show that the MWIS problem can be efficiently solved in the class of $(S_{1,2,2}, S_{1,1,3}, \text{co-chair})$ -free graphs by analyzing the atomic structure of the subclasses of this class of graphs.

5.1 $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, \text{gem})$ -free graphs

Theorem 9. *The MWIS problem can be solved in polynomial time for $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, \text{gem})$ -free graphs.*

Proof. Let G be an $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, \text{gem})$ -free graph. First suppose that G is prime. Then by Lemma 3, G is diamond-free. Since the MWIS problem in $(S_{1,2,2}, S_{1,1,3}, \text{diamond})$ -free graphs can be solved in polynomial time (by Theorem 8), MWIS can be solved in polynomial time for G , by Theorem 1. Then the time complexity is the same when G is not prime, by Theorem 1. \square

5.2 $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, H^*)$ -free graphs

Theorem 10. *Let $G = (V, E)$ be a prime $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, H^*)$ -free graph. Then every atom of G is nearly gem-free (see Figure 3 for the graph H^*).*

Proof. Let G' be an atom of G . We want to show that G' is nearly gem-free, so let us assume on the contrary that there is a vertex $v \in V(G')$ such that the anti-neighborhood of v in G' contains an induced gem H . Let H have vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_1v_5, v_2v_5, v_3v_5, v_4v_5\}$. For $i \in \{1, 2, \dots, 5\}$, we define sets A_i , A_i^+ , A_i^- , and Q , with

respect to G , v and H , as in the last paragraph of Section 2. Then we have the following properties:

Claim 10.1. *Every vertex x in $N(H)$ satisfies either $v_5 \in N(x)$ or x has at least one neighbor in $\{v_2, v_3\}$. In particular, (i) if $x \in A_1$, then $N_H(x) = \{v_5\}$, and (ii) if $x \in A_2$, then $N_A(x) \in \{\{v_2, v_3\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_1, v_3\}, \{v_2, v_4\}\}$.*

Proof of Claim 10.1. Suppose to the contrary that $xv_5 \notin E$ and x has no neighbor in $\{v_2, v_3\}$. Then, up to symmetry, we have $xv_1 \in E$, and so $\{x, v_1, v_2, v_3, v_5\}$ induces a co-chair in G , a contradiction. \diamond

Let B^* denote the set $\{x \in A_2 : N_A(x) = \{v_1, v_3\}\} \cup \{x \in A_2 : N_A(x) = \{v_2, v_4\}\}$.

Claim 10.2. $A_2^+ = A_3^+ = A_4^+ = \emptyset$.

Proof of Claim 10.2.

Assume the contrary and let $x \in A_2^+ \cup A_3^+ \cup A_4^+$. There is a vertex z in Q such that $xz \in E$. First suppose that $xv_5 \notin E$. Then by Claim 10.1, x has at least one neighbor in $\{v_2, v_3\}$. Now, if $\{v_2, v_3\} \subseteq N_H(x)$, then $\{v_5, v_2, v_3, x, z\}$ induces a co-chair in G , which is a contradiction. So, we may assume that x has exactly one neighbor in $\{v_2, v_3\}$, say $xv_2 \in E$ and $xv_3 \notin E$. Since $x \in A_i^+$ ($i \geq 2$), either $xv_1 \in E$ or $xv_4 \in E$. But, then either $\{v_5, v_1, v_2, x, z\}$ induces a co-chair in G , or $\{v_5, v_2, v_3, v_4, x, z\}$ induces a H^* in G , respectively, a contradiction. So, suppose that $xv_5 \in E$. Then it follows that there is a clique $\{p, q, r\} \subset V(H)$ such that $xp, xq \in E$ and $xr \notin E$. Then $\{z, x, p, q, r\}$ induces a co-chair in G , a contradiction. \diamond

Claim 10.3. *For each $i \in \{2, \dots, 5\}$, A_5^+ is complete to A_i^- .*

Proof of Claim 10.3. Assume the contrary. Let $x \in A_5^+$ and $y \in A_i^-$ be such that $xy \notin E$. Since $x \in A_5^+$, there exists $z \in Q$ such that $xz \in E$. Now, if $y \in (\bigcup_{i=2}^5 A_i^-) \setminus B^*$, then there exist vertices $p, q \in V(H)$ such that $pq \in E$ and $yp, yq \in E$. Then $\{z, x, p, q, y\}$ induces a co-chair in G , a contradiction. So, $y \in B^*$. But, now, $\{x, v_4, v_5, v_1, y\}$ induces a co-chair in G , a contradiction. \diamond

Claim 10.4. A_5^+ is a clique.

Proof of Claim 10.4: Suppose the contrary. Then $G[A_5^+]$ has a co-connected component X of size at least 2. Since G is prime, X is not a module in G , so there is a vertex z in $V(G) \setminus X$ that distinguishes two vertices x and y of X , and since X is co-connected we can choose x and y non-adjacent. Clearly $z \notin H$ and $z \notin A_5^+$. So either (i) z has no neighbor in H , or (ii)

$z \in A^-$ and so, by Claim 10.3 (since $\{z\}$ is not complete to A_5^+), $z \in A_1^-$, or (iii) $z \in A^+$ and so, by Claim 10.2, $z \in A_1^+$. In either of these three cases, by Claim 10.1, we see that $\{z, x, y, v_1, v_2\}$ induces a co-chair in G , a contradiction. \diamond

Let $B = A_2^- \cup A_3^- \cup A_4^- \cup A_5^-$.

Claim 10.5. *If $A_1^+ \neq \emptyset$, then $\{v_5\}$ is complete to B .*

Proof of Claim 10.5: Assume on the contrary that there is a vertex $x \in B$ such that $xv_5 \notin E(G)$. Since $A_1^+ \neq \emptyset$, there is a vertex $a \in A_1^+$ and a vertex $z \in Q$ such that $az \in E(G)$. Recall that $N_H(a) = \{v_5\}$, by Claim 10.1. Now, if there exists vertices $p, q \in \{v_1, v_2, v_3, v_4\}$ such that $pq \in E$ and $xp, xq \in E$, then since $\{x, p, q, v_5, a\}$ does not induce a co-chair in G , $xa \in E$. But, then $\{z, a, x, v_5, p, q\}$ induces a H^* in G , a contradiction. So, we may assume that $N_H(x) \cap \{v_1, v_2, v_3, v_4\}$ is an independent set. Hence by Claim 10.1, $N_H(x)$ is either $\{v_1, v_3\}$ or $\{v_2, v_4\}$. We may assume, up to symmetry, that $N_H(x) = \{v_1, v_3\}$. Then since $\{z, a, v_5, v_1, x, v_4\}$ does not induce an $S_{1,2,2}$ in G , $xa \in E$. But, then $\{z, a, x, v_3, v_2, v_4\}$ induces an $S_{1,1,3}$ in G , a contradiction. \diamond

Claim 10.6. *There is no edge between A_1^+ and B .*

Proof of Claim 10.6: Assume the contrary, and let $a \in A_1^+$ and $b \in B$ be such that $ab \in E$. Since $a \in A_1^+$, there exists $y \in Q$ such that $ay \in E$. Since $b \in B$, by Claim 10.5 there exists an index j ($j \in \{1, \dots, 4\}$) such that $bv_5, bv_j \in E$. Then $\{y, a, b, v_5, v_j\}$ induces a co-chair in G , which is a contradiction. \diamond

Claim 10.7. *If $a \in A_1^+$, $b \in B$, and $x \in A_1^-$, then $\{a, b, x\}$ does not induce a path in G .*

Proof of Claim 10.7: Assume the contrary. Since $a \in A_1^+$, there exists a vertex $z \in Q$ such that $az \in E$. By Claim 10.6, $ab \notin E$. Thus, by the assumption, $ax \in E$ and $xb \in E$. Also, by Claims 10.1 and 10.5, we have $av_5, xv_5 \in E$ and $bv_5 \in E$. But, now $\{z, a, x, b, v_5\}$ induces a co-chair in G , which is a contradiction. \diamond

Suppose that $A_1^+ = \emptyset$. Then $A^+ = A_5^+$, which is a clique by Claim 10.4. Since A^+ is a separator in G between H and Q , it follows that $V(G') \cap A^+$ is a clique separator in G' between H and $V(G') \cap Q$ (which contains v); this is a contradiction to the fact that G' is an atom. Therefore $A_1^+ \neq \emptyset$. Now, Claims 10.1 and 10.2 imply that $N(v_4) \subseteq \{v_3, v_5\} \cup B \cup A_5^+$, and Claims 10.4, 10.6, and 10.7 imply that $A_5^+ \cup \{v_2, v_3, v_5\}$ is a clique separator between $\{v_4\}$

and Q in G . Hence $V(G') \cap (A_5^+ \cup \{v_2, v_3, v_5\})$ is a clique separator between $\{v_4\}$ and $V(G') \cap Q$ in G' , again a contradiction to the fact that G' is an atom. \square

Theorem 11. *The MWIS problem can be solved in polynomial time for $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, H^*)$ -free graphs.*

Proof. Let G be an $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, H^*)$ -free graph. First suppose that G is prime. By Theorem 10, every atom of G is nearly gem-free. Since the MWIS in $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, \text{gem})$ -free graphs can be solved in polynomial time (by Theorem 9), MWIS in $(S_{1,2,2}, S_{1,1,3}, \text{co-chair}, H^*)$ -free graphs can be solved in polynomial time, by Theorem 2. Then the time complexity is the same when G is not prime, by Theorem 1. \square

5.3 $(S_{1,2,2}, S_{1,1,3}, \text{co-chair})$ -free graphs

Theorem 12. *Let $G = (V, E)$ be a prime $(S_{1,2,2}, S_{1,1,3}, \text{co-chair})$ -free graph. Then every atom of G is nearly H^* -free.*

Proof. Let G' be an atom of G . We want to show that G' is nearly H^* -free, so let us assume on the contrary that there is a vertex $v \in V(G')$ such that the anti-neighborhood of v in G' contains an induced H^* as shown in Figure 3. For $i = 1, \dots, 6$ we define sets A_i , A_i^+ , A_i^- , and Q , with respect to G , v and H , as in the last paragraph of Section 2. Then we have the following properties:

Claim 12.1. $A_1 = \emptyset$.

Proof of Claim 12.1. Suppose to the contrary that $A_1 \neq \emptyset$, and let $x \in A_1$. Then: (i) If $N_{H^*}(x)$ is either $\{v_1\}$ or $\{v_3\}$, then $\{x\} \cup V(H^*)$ induces a graph which is isomorphic to H_7 in G , a contradiction to Lemma 1. (ii) If $N_{H^*}(x)$ is either $\{v_2\}$ or $\{v_4\}$, then $\{x, v_1, v_2, v_3, v_4\}$ induces a co-chair in G , which is a contradiction. (iii) If $N_{H^*}(x) = \{v_5\}$, then $\{x\} \cup V(H^*)$ induces a graph which is isomorphic to H_6 in G , a contradiction to Lemma 1. (iv) If $N_{H^*}(x) = \{v_6\}$, then $\{x\} \cup V(H^*)$ induces a graph which is isomorphic to H_4 in G , a contradiction to Lemma 1. So, the claim holds. \diamond

Claim 12.2. *If $x \in A_2$, then $N_{H^*}(x) \in \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_5, v_6\}\}$.*

Proof of Claim 12.2. For, otherwise if $N_{H^*}(x) \in \{\{v_1, v_3\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_4, v_5\}, \{v_4, v_6\}\}$, then $\{x, v_1, v_2, v_3, v_4\}$ induces a co-chair in G , which is a

contradiction, and if $N_{H^*}(x)$ is $\{v_2, v_4\}$, then $\{x\} \cup V(H^*)$ induces a graph which is isomorphic to H_5 in G , a contradiction to Lemma 1. So, the claim holds. \diamond

Claim 12.3. $A_2^+ = \emptyset$.

Proof of Claim 12.3. Suppose to the contrary that $A_2^+ \neq \emptyset$, and let $x \in A_2^+$. Then there is a vertex z in Q such that $xz \in E$. We use Claim 12.2 to derive a contradiction to our assumption as follows: Now, if $N_{H^*}(x) \in \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\}$, then it follows that there is a clique $\{p, q, r\} \subset V(H^*)$ such that $xp, xq \in E$ and $xr \notin E$. But, then $\{z, x, p, q, r\}$ induces a co-chair in G , which is a contradiction. Next, if $N_{H^*}(x) \in \{\{v_1, v_5\}, \{v_1, v_6\}, \{v_3, v_5\}, \{v_3, v_6\}\}$, then $\{z, x, v_1, v_3, v_5, v_6\}$ induces an $S_{1,2,2}$ in G , which is a contradiction. Finally, if $N_{H^*}(x)$ is $\{v_5, v_6\}$, then $\{v_1, v_2, v_3, v_4, v_5, x, z\}$ induces a graph which is isomorphic to H_4 in G , a contradiction to Lemma 1. So, $A_2^+ = \emptyset$, and the claim holds. \diamond

Claim 12.4. If $x \in A_3$, then $N_{H^*}(x) \in \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_1, v_5, v_6\}, \{v_2, v_3, v_4\}, \{v_2, v_4, v_6\}, \{v_3, v_5, v_6\}\}$.

Proof of Claim 12.4. Suppose the contrary. Now, if $v_6 \in N_{H^*}(x)$, then since $x \in A_3$, $|N_{H^*}(x) \cap \{v_1, v_2, v_3, v_4\}| \in \{1, 2\}$. If $|N_{H^*}(x) \cap \{v_1, v_2, v_3, v_4\}| = 2$, then it follows that there is a clique $\{p, q, r\} \subset \{v_1, v_2, v_3, v_4\}$ such that $xp, xq \in E$ and $xr \notin E$. But, then $\{v_6, x, p, q, r\}$ induces a co-chair in G , which is a contradiction. So, $|N_{H^*}(x) \cap \{v_1, v_2, v_3, v_4\}| = 1$, and hence $v_5 \in N_{H^*}(x)$. Since $x \in A_3$ and by our contrary assumption, either $v_2 \in N_{H^*}(x)$ or $v_4 \in N_{H^*}(x)$. But, then $\{v_1, v_2, v_3, v_4, x\}$ induces a co-chair in G , which is a contradiction. So, we may assume that $v_6 \notin N_{H^*}(x)$. Now, (i) if $N_{H^*}(x)$ is $\{v_1, v_2, v_3\}$, then $\{x, v_1, v_3, v_4, v_5\}$ induces a co-chair in G , (ii) if $N_{H^*}(x)$ is $\{v_2, v_3, v_5\}$, then $\{x, v_2, v_3, v_5, v_6\}$ induces a co-chair in G , and (iii) if $N_{H^*}(x)$ is $\{v_1, v_2, v_5\}$, then $\{x, v_1, v_2, v_5, v_6\}$ induces a co-chair in G , which are contradictions. Finally, if $N_{H^*}(x)$ is $\{v_2, v_4, v_5\}$, then $\{x\} \cup V(H^*)$ induces a graph which is isomorphic to H_1 in G , a contradiction to Lemma 1. Hence the claim is proved. \diamond

Claim 12.5. If $x \in A_3^+$, then $N_{H^*}(x)$ is either $\{v_1, v_5, v_6\}$ or $\{v_3, v_5, v_6\}$.

Proof of Claim 12.5. For, otherwise, by Claim 12.4, $N_{H^*}(x) \in \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_4, v_6\}\}$. Since $x \in A_3^+$, there is a vertex z in Q such that $xz \in E$. Now, if $N_{H^*}(x) \in \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_2, v_3, v_4\}\}$, then it follows that there is a clique $\{p, q, r\} \subset V(H^*)$ such that $xp, xq \in E$ and $xr \notin E$. But, then $\{z, x, p, q, r\}$ induces a co-chair in G , which is

a contradiction. Next, if $N_{H^*}(x)$ is $\{v_2, v_4, v_6\}$, then $\{v_1, v_2, v_3, v_4, v_6, x, z\}$ induces a graph which is isomorphic to H_6 in G , a contradiction to Lemma 1. So the claim is proved. \diamond

Let B'_3 denotes the set $\{x \in A_3^+ \mid N_{H^*}(x) = \{v_1, v_5, v_6\}\}$ and let B''_3 denotes the set $\{x \in A_3^+ \mid N_{H^*}(x) = \{v_3, v_5, v_6\}\}$.

Claim 12.6. B'_3 and B''_3 are cliques in G .

Proof of Claim 12.6. Suppose to the contrary that there exists vertices $x, y \in B'_3$ such that $xy \notin E$. Since $x \in A_3^+$, there exists a vertex z in Q such that $xz \in E$. Now, if $yz \in E$, then $\{z, x, y, v_5, v_6, v_1, v_3\}$ induces a graph which is isomorphic to H_5 in G , which contradicts Lemma 1, and if $yz \notin E$, then $\{v_5, v_6, x, y, z\}$ induces a co-chair in G , which is a contradiction. Hence, B'_3 is a clique in G . Similarly, B''_3 is also a clique in G . \diamond

Claim 12.7. At most one of B'_3 or B''_3 is non-empty.

Proof of Claim 12.7. Suppose the contrary, and let $x \in B'_3$ and $y \in B''_3$. Then since $\{x, y, v_1, v_5, v_6\}$ does not induce a co-chair in G , $xy \in E$. Since $x \in A_3^+$, there exists a vertex z in Q such that $xz \in E$. Now, if $yz \in E$, then $\{v_2, v_5, x, y, z\}$ induces a co-chair in G , which is a contradiction, and if $yz \notin E$, then $\{z, x, y, v_3, v_2, v_4\}$ induces an $S_{1,1,3}$ in G , which is a contradiction. Hence the claim. \diamond

Claim 12.8. $A_4^+ = \emptyset$.

Proof of Claim 12.8. Suppose to the contrary that $A_4^+ \neq \emptyset$ and let $x \in A_4^+$. There is a vertex z in Q such that $xz \in E$. Now, if x is adjacent to all the vertices in $\{v_1, v_2, v_3, v_4\}$, then $\{z, x, v_1, v_2, v_4, v_4, v_6\}$ induces a graph which is isomorphic to H_7 in G , a contradiction to Lemma 1. So, we may assume that x is non-adjacent to at least one vertex in $\{v_1, v_2, v_3, v_4\}$. Also, since $x \in A_4^+$, x is adjacent to at least two vertices in $\{v_1, v_2, v_3, v_4\}$. Now, if $N_{H^*}(x)$ is $\{v_2, v_4, v_5, v_6\}$, then $\{x, v_1, v_2, v_5, v_6\}$ induces a co-chair in G , a contradiction, and in all the other cases, there is a clique $\{p, q, r\} \subset V(H^*)$ such that $xp, xq \in E$ and $xr \notin E$. But, then $\{z, x, p, q, r\}$ induces a co-chair in G , which is a contradiction. \diamond

Claim 12.9. $A_5^+ = \emptyset$.

Proof of Claim 12.9. Suppose to the contrary that $A_5^+ \neq \emptyset$ and let $x \in A_5^+$. Then there is a vertex z in Q such that $xz \in E$. Suppose x is adjacent to all the vertices in $\{v_1, v_2, v_3, v_4\}$. Further, if x is adjacent to v_5 , then

$\{x, v_2, v_3, v_5, v_6\}$ induces a co-chair in G , which is a contradiction, and if x is adjacent to v_6 , then $\{x\} \cup V(H^*)$ induces a graph which is isomorphic to H_2 in G , a contradiction to Lemma 1. So, we may assume that x is non-adjacent to exactly one vertex in $\{v_1, v_2, v_3, v_4\}$. Then, there is a clique $\{p, q, r\} \subset V(H^*)$ such that $xp, xq \in E$ and $xr \notin E$. But, then $\{z, x, p, q, r\}$ induces a co-chair in G , which is a contradiction. \diamond

Claim 12.10. A_3^+ is complete to A_6^+ .

Proof of Claim 12.10. Suppose to the contrary that there exist vertices $x \in A_3^+$ and $y \in A_6^+$ such that $xy \notin E$. Then by Claim 12.5, $N_{H^*}(x)$ is either $\{v_1, v_5, v_6\}$ or $\{v_3, v_5, v_6\}$. Then either $\{v_3, y, v_5, v_6, x\}$ or $\{v_1, y, v_5, v_6, x\}$ induces a co-chair in G , a contradiction. \diamond

Claim 12.11. For each $i \in \{2, \dots, 6\}$, A_6^+ is complete to A_i^- .

Proof of Claim 12.11. Assume the contrary. Let $x \in A_6^+$ and $y \in A_i^-$ be such that $xy \notin E$. Since $x \in A_6^+$, there exists $z \in Q$ such that $xz \in E$. Now, if there exists vertices $p, q \in V(H^*)$ such that $pq \in E$ and $py, qy \in E$, then $\{z, x, p, q, y\}$ induces a co-chair in G , which is a contradiction. So, we assume that $N_{H^*}(y)$ is an independent set. Then by the above claims on A_i^+ , $i \geq 2$, we have $N_{H^*}(y) \in \{\{v_1, v_5\}, \{v_1, v_6\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_2, v_4, v_6\}\}$. Now, if $N_{H^*}(y) \in \{\{v_1, v_5\}, \{v_3, v_5\}\}$, then $\{z, x, y\} \cup V(H^*)$ induces a graph which is isomorphic to H_8 in G , a contradiction to Lemma 1. So, $N_{H^*}(y) \in \{\{v_1, v_6\}, \{v_3, v_6\}, \{v_2, v_4, v_6\}\}$. But, then $\{v_1, v_4, v_5, x, y\}$ induces a co-chair in G (if $N_{H^*}(y) = \{v_1, v_6\}$), and $\{v_3, v_4, v_5, x, y\}$ induces a co-chair in G (if $N_{H^*}(y) = \{v_3, v_6\}$), which are contradictions. Finally, if $N_{H^*}(y) = \{v_2, v_4, v_6\}$, then $\{z, x, y\} \cup V(H^*)$ induces a graph which is isomorphic to H_3 in G , a contradiction to Lemma 1. \diamond

Claim 12.12. A_6^+ is a clique.

Proof of Claim 12.12: Suppose the contrary. Then $G[A_6^+]$ has a co-connected component X of size at least 2. Since G is prime, X is not a non-trivial module in G , so there is a vertex z in $V(G) \setminus X$ that distinguishes two vertices x and y of X , and since X is co-connected we can choose x and y non-adjacent. We may assume (wlog.) that $xz \in E$ and $yz \notin E$. Clearly $z \notin V(H^*)$ and $z \notin A_6^+$. By Claims 12.1 and 12.11, $z \notin A^-$, and by Claims 12.1, 12.3, 12.8, 12.9, and 12.10, we have $z \notin A^+$. Hence, z has no neighbor in H^* , and we see that $\{z, x, y, v_1, v_2\}$ induces a co-chair in G , a contradiction. \diamond

By Claims 12.1, 12.3, 12.8, 12.9, we have $A^+ = A_3^+ \cup A_6^+$, which is a clique by Claims 12.6, 12.7, 12.10, and 12.12. Since A^+ is a separator in G between H^* and Q , it follows that $V(G') \cap A^+$ is a clique separator in G' between H^* and $V(G') \cap Q$ (which contains v); this is a contradiction to the fact that G' is an atom. \square

Theorem 13. *The MWIS problem can be solved in polynomial time for $(S_{1,2,2}, S_{1,1,3}, \text{co-chair})$ -free graphs.*

Proof. Let G be an $(S_{1,2,2}, S_{1,1,3}, \text{co-chair})$ -free graph. First suppose that G is prime. By Theorem 12, every atom of G is nearly H^* -free. Since the MWIS problem in $(S_{1,2,2}, S_{1,1,3}, H^*, \text{co-chair})$ -free graphs can be solved in polynomial time (by Theorem 11), MWIS can be solved in polynomial time for G , by Theorem 2. Then the time complexity is the same when G is not prime, by Theorem 1. \square

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